Discontinuities and singularities in the structure of the most tight trefoil knot

Sylwester Przybyl
Poznan University of Technology, Poland

The appropriately modified Finite Element Method has been used to find the most tight conformation of the trefoil knot tied on the perfect rope. The processed knot contains $N = 200640$ vertices. Each vertex is connected via longitudinally elastic beams with other vertices. The forces with which the beams act on the vertices shift them slowly in such a way that in the final conformation all forces acting on each of the vertices sum up to zero. The overlapping were prevented and curvature was not allowed to exceed $1/R$. Numerical calculations simulating the tightening of the knot lasted on a PC computer a few months. The final knot is equilateral. Its segment length $dl = 0.000\ 163\ 192\ 456\ 437 \pm 2 \cdot 10^{-15}$ and its total ropelenght $L = 32.742\ 934\ 477 \pm 1 \cdot 10^{-9}$.

The final knot contains 6 pieces, where curvature reaches its highest allowable value. The length of the pieces $l_{1,2} = 0.161$. The pieces contain almost 1000 verices.

Two, well supported by the numerical data, conjectures are most essential:

*1. **Curvature of the ideal trefoil knot is not continuous**. It reaches the maximum allowable value* $\kappa = 1$ *on six finite intervals. The plateaus of maximum curvature are separated at both sides by discontinuities.*

*2. **The torsion of the ideal trefoil is not a conventional function**: in points, where curvature displays primary discontinuities, torsion displays Dirac delta components.*

The values describing the knot are:

$L = 32.742\ 934\ 477 \pm 1 \cdot 10^{-9}$,
$l_1 = 0.158 \pm 0.001$,
$l_2 = 0.319 \pm 0.001$,
$l_{1,2} = l_2 - l_1 = 0.161 \pm 0.002$,

$\Delta\kappa_1 = 0.264 \pm 0.005$,
$\Delta\kappa_2 = 0.115 \pm 0.005$,
$\Delta\alpha_1 = 0.159 \pm 0.005$,
$\Delta\alpha_2 = 0.062 \pm 0.005$.

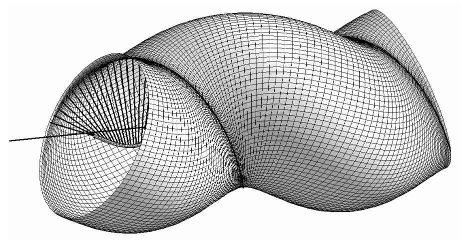

Fig. 1. An example of the **curvature limited helix** . The perfect rope of radius R = 1 has been formed into a helix. The radius of the helix r_H = 0.2, while its pitch P_H = 2.513 27. At such parameter values the curvature of the helix is equal 1. Points of contacts of type II draw a helical line on the surface of the rope. The contact points coincide with the centers of the osculating circles.

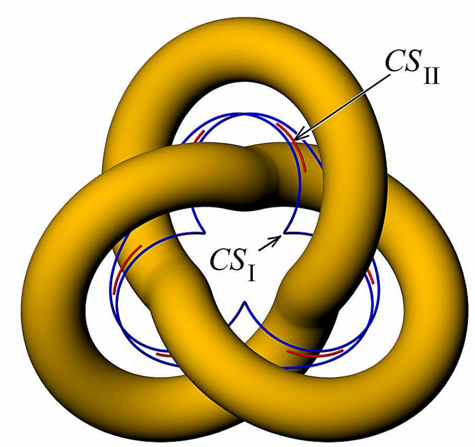

Fig. 2. Location of **the contact sets CS_I and CS_{II}** within the ideal trefoil. Set CS_I of contact points of the I kind is connected and knotted. Set CS_{II} of the contact points of the second kind consists of six congruent pieces. The arrow indicates one of them.

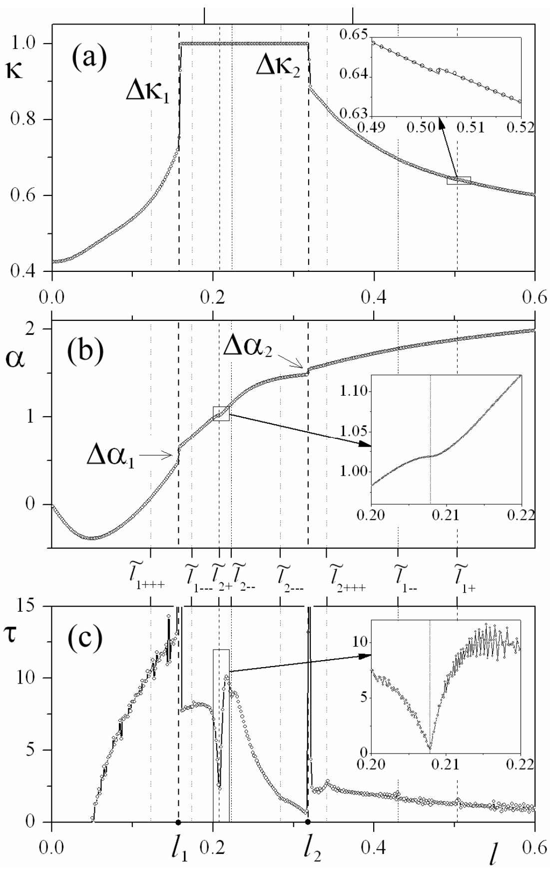

Fig. 3**. Curvature (a), accumulated torsion (b) and torsion (c)** of the most tight trefoil knot in the initial part of the [0, *L*/6) interval. The curvature plot is clearly discontinuous at l_1 and l_2. The torsion plot displays sharp and high peaks (their upper parts are not visible). As we guess, the peaks turn within the ideal trefoil knot into Dirac deltas. Note that values of torsion at both sides of the peaks are different.

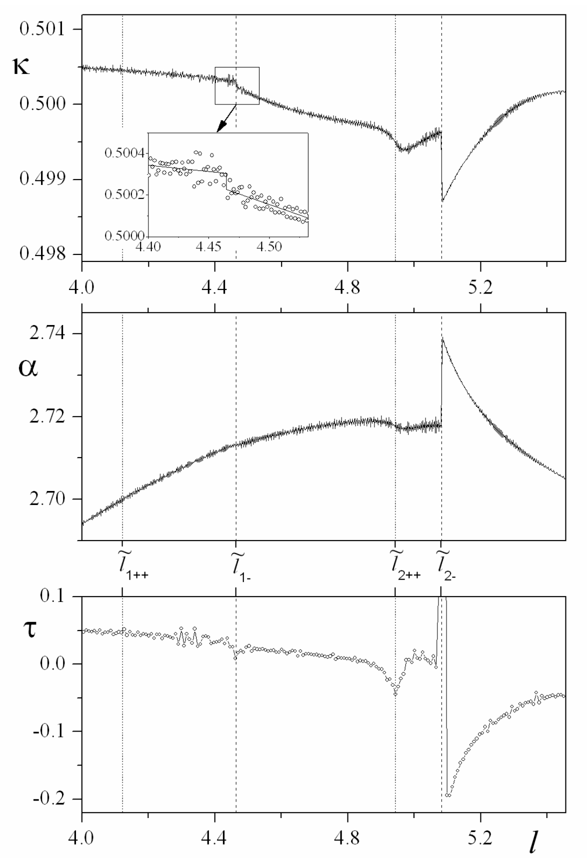

Fig. 4. Enlarged plots **of curvature (a), accumulated torsion (b) and torsion(c)**, in the end part of the [0, L/6) interval. The secondary discontinuity of curvature visible at $\tilde{l}_{2-}$ is induced via contacts by the primary discontinuity shown in figure 3(a).

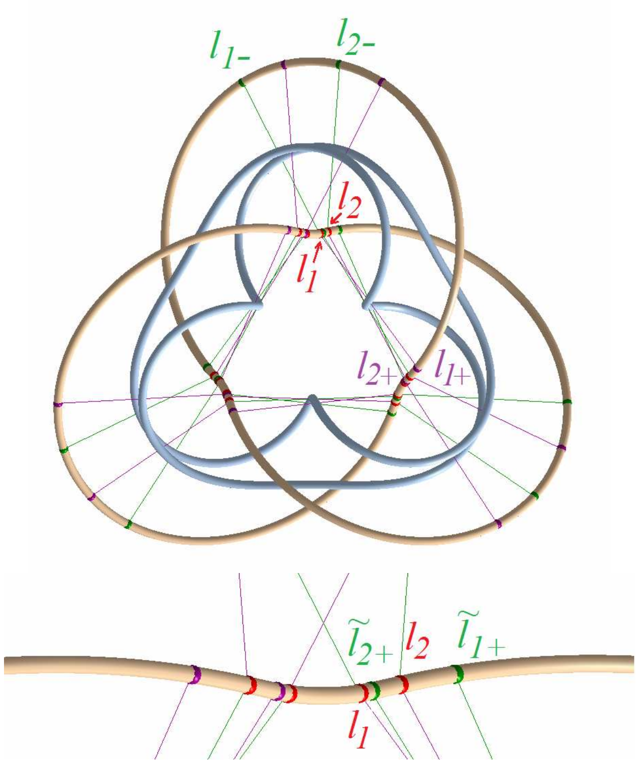

Fig. 5. The locations l_1 and l_2 of the primary discontinuities of curvature. Via contacts the discontinuities induce at $\tilde{l}_{1+}, \tilde{l}_{1-}, \tilde{l}_{2+}, \tilde{l}_{2-}$ the secondary singularities in curvature and/or torsion.

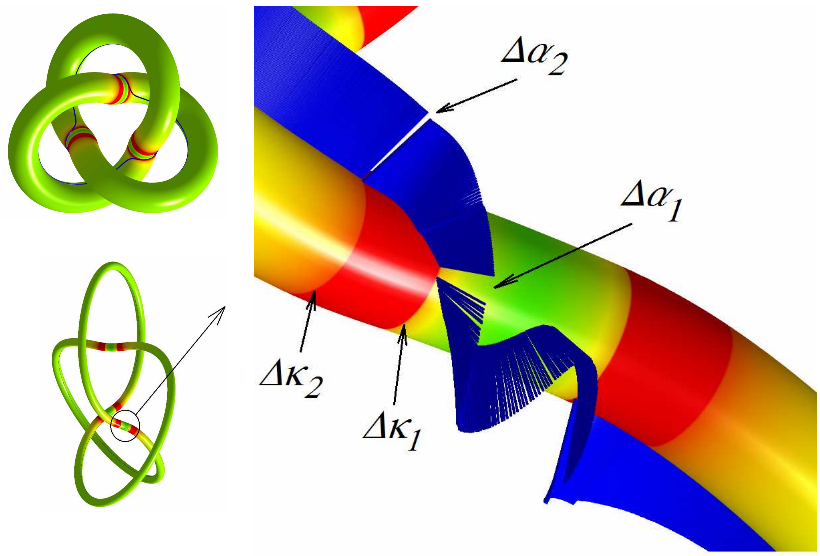

Fig. 6. The ribbon of the normal vectors in the region of the primary discontinuities.

REFERENCES

S. Przybyl and P. Pieranski, High resolution portrait of the ideal trefoil knot, J. Phys. A: Math. Theor. 47 (2014)